Progress on the Strong Eshelby's Conjecture and Extremal Structures for the Elastic Moment Tensor[*]

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Abstract

We make progress towards proving the strong Eshelby's conjecture in three dimensions. We prove that if for a single nonzero uniform loading the strain inside inclusion is constant and further the eigenvalues of this strain are either all the same or all distinct, then the inclusion must be of ellipsoidal shape. As a consequence, we show that for two linearly independent loadings the strains inside the inclusions are uniform, then the inclusion must be of ellipsoidal shape. We then use this result to address a problem of determining the shape of an inclusion when the elastic moment tensor (elastic polarizability tensor) is extremal. We show that the shape of inclusions, for which the lower Hashin-Shtrikman bound either on the bulk part or on the shear part of the elastic moment tensor is attained, is an ellipse in two dimensions and an ellipsoid in three dimensions.

1 Introduction and statements of results

In theory of composites or micro-structures, it is important to find inclusion shapes which produce the minimal energy. In relation to such shapes Eshelby [13] showed that if the inclusion is of ellipsoidal shape, then for any uniform loading the strain inside Ω is uniform. We call this remarkable property Eshelby's uniformity property.

Eshelby then conjectured in [14] that ellipsoids are the only shape (structure) with such a uniformity property.

Eshelby's conjecture may be interpreted in two different ways:

Weak Eshelby's conjecture. If the strain is constant inside Ω for all loadings, then Ω is an ellipse (2D) or an ellipsoid (3D).

Strong Eshelby's conjecture. If the strain is constant inside Ω for a single loading, then Ω is an ellipse (2D) or an ellipsoid (3D).

The strong Eshelby conjecture of course implies the weak one.

The strong Eshelby's conjecture has been proved to be true in two dimensions by Sendeckyj [28] (see also [19, 22] for alternative proofs). However it is only recently that the weak Eshelby conjecture was proved to be true in three dimensions: by Kang-Milton [19] and Liu [22]. We refer to above mentioned papers (and [15]) for comprehensive account of developments on the Eshelby conjecture.

Regarding the strong Eshelby conjecture in three dimensions, important pro–gress has been made by Liu. He showed in [22] that the conductivity version of the strong Eshelby conjecture fails to be true completely (see [22] for a precise statements.) However, the strong Eshelby's conjecture (for elasticity) has not been proved or disproved. In this paper we consider the strong Eshelby conjecture. Even though we are not able to resolve the conjecture completely, we obtain results which is stronger than the weak version of Eshelby's conjecture (and weaker than the strong version). We show that if the strain inside inclusion is constant and in addition the eigenvalues of the constant strain are either all the same or all distinct, then the inclusion is of ellipsoidal shape. We then use this result to show that for two linearly independent loadings the strains inside the inclusions are uniform, then the inclusion must be of ellipsoidal shape. It is worth emphasizing that the weak Eshelby's conjecture requires 6 linearly independent loadings while the strong Eshelby's conjecture does a single loadings.

In order to present results in more precise way let us introduce some notation. Let Ω be a bounded domain with a Lipschitz boundary in $\mathbb{R}^d$, $d = 2, 3$. The domain Ω is occupied by a homogeneous isotropic elastic material whose Lamé parameters are $\tilde{\lambda}$ and $\tilde{\mu}$. We assume that the background (the matrix) is also homogeneous and isotropic, and its Lamé parameters are λ and μ. Then the elasticity tensors for the matrix and the inclusion can be written respectively as

$$\mathbb{C}^0 := \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbb{I} \quad \text{and} \quad \mathbb{C}^1 := \tilde{\lambda} \mathbf{I} \otimes \mathbf{I} + 2\tilde{\mu} \mathbb{I}, \tag{1.1}$$

where $\mathbf{I}$ is the $d \times d$ identity matrix (2-tensor) and $\mathbb{I}$ is the identity 4-tensor. The the elasticity tensor for $\mathbb{R}^d$ in the presence of the inclusion Ω is then given by

$$\mathbb{C}_\Omega := (1 - 1_\Omega)\mathbb{C}^0 + 1_\Omega \mathbb{C}^1, \tag{1.2}$$

where 1_Ω is the indicator function of Ω.

Let κ and $\tilde{\kappa}$ be bulk moduli of $\mathbb{R}^d \setminus \overline{\Omega}$ and Ω, respectively, namely,

$$\kappa = d\lambda + 2\mu \quad \text{and} \quad \tilde{\kappa} = d\tilde{\lambda} + 2\tilde{\mu}, \quad d = 2, 3.$$

It is always assumed that the strong convexity condition holds, *i.e.*,

$$\mu > 0, \quad \kappa > 0, \quad \tilde{\mu} > 0 \quad \text{and} \quad \tilde{\kappa} > 0 . \tag{1.3}$$

We also assume that

$$(\lambda - \tilde{\lambda})(\mu - \tilde{\mu}) > 0 ,$$

which implies that $\mathbb{C}^1 - \mathbb{C}^0$ is either positive or negative definite as an operator on the space M_d^s of all $d \times d$ symmetric matrices.

We consider the following problem of the Lamé system of linear elasticity: For a given non-zero symmetric $d \times d$ matrix $\mathbf{A}$

$$\begin{cases} \nabla \cdot \mathbb{C}_\Omega \mathcal{E}(\mathbf{u}) = 0 & \text{in } \mathbb{R}^d \\ \mathbf{u}(\mathbf{x}) - \mathbf{A}\mathbf{x} = O(|\mathbf{x}|^{1-d}) & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{1.4}$$

where $\mathcal{E}(\mathbf{u})$ is the strain tensor, *i.e.*,

$$\mathcal{E}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \quad (T \text{ for transpose}).$$

The matrix $\mathbf{A}$ represents a uniform loading at infinity.

In this paper we prove the following improvements of the weak Eshelby conjecture for the three dimensional elasticity.

Theorem 1.1 *Let Ω be a simply connected bounded domain in $\mathbb{R}^3$ with a Lipschitz boundary. If the strain tensor $\mathcal{E}(\mathbf{u})$ of the solution $\mathbf{u}$ to (1.4) is constant in Ω for a nonzero symmetric matrix $\mathbf{A}$ and $\mathcal{E}(\mathbf{u})$ within Ω has eigenvalues which are either all distinct or all the same, then Ω is an ellipsoid.*

Theorem 1.2 *Let Ω be a simply connected bounded domain in $\mathbb{R}^3$ with a Lipschitz boundary. If the strain tensors of solutions to (1.4) for two linearly independent $\mathbf{A}$'s are constant in Ω, then Ω is an ellipsoid.*

The second main result of this paper is on the shape of the inclusion whose elastic moment tensor (elastic polarizability tensor) has an extremal property. In order to explain the second result, we take the following definition of the Elastic Moment Tensor (henceforth denoted as the EMT) [5, Lemma 10.3]: Let $\mathbf{A}$ be a $d \times d$ matrix and let $\mathbf{u_A}$ be the solution to (1.4) corresponding to $\mathbf{A}$. Then the EMT $\mathbb{M}$ associated with the inclusion Ω and the elasticity tensors $\mathbb{C}^0$ and $\mathbb{C}^1$ is a 4-tensor defined by

$$\mathbb{M}\mathbf{A} = \int_\Omega (\mathbb{C}^1 - \mathbb{C}^0)\mathcal{E}(\mathbf{u_A}) \, d\mathbf{x}. \tag{1.5}$$

The EMT may be defined in many different but equivalent ways. It is worth noticing that if the strain $\mathcal{E}(\mathbf{u_A}) = \mathbf{B}$ is constant in Ω, then

$$\mathbb{M}\mathbf{A} = |\Omega|(\mathbb{C}^1 - \mathbb{C}^0)\mathbf{B}, \tag{1.6}$$

where $|\Omega|$ denotes the volume of Ω.

The EMT enjoys several important properties. For example, it is symmetric and positive-definite or negative-definite on the space M_d^s of $d \times d$ symmetric matrices, depending on the sign of $\tilde{\mu} - \mu$. The notion of EMT is being used in variety of contexts such as detection of small elastic inclusions for non-destructive evaluation and medical imaging [1, 2, 3, 7, 5, 16, 17] and effective medium theory [5, 6, 23].

Let us introduce more notation in order to recall the optimal trace bounds (the Hashin-Shtrikman bounds) for the EMT. Let

$$\mathbf{\Lambda}_1 := \frac{1}{d}\mathbf{I} \otimes \mathbf{I}, \quad \mathbf{\Lambda}_2 := \mathbb{I} - \mathbf{\Lambda}_1.$$

Then the elasticity tensor $\mathbb{C}^0$ may be written as

$$\mathbb{C}^0 = d\kappa\mathbf{\Lambda_1} + 2\mu\mathbf{\Lambda_2},$$

and likewise for $\mathbb{C}^1$. Since for any $d \times d$ symmetric matrix $\mathbf{A}$, $\mathbf{I} \otimes \mathbf{I}(\mathbf{A}) = \operatorname{tr}(\mathbf{A})\mathbf{I}$ and $\mathbb{I}(\mathbf{A}) = \mathbf{A}$, one can immediately see that

$$\mathbf{\Lambda}_1\mathbf{\Lambda}_1 = \mathbf{\Lambda}_1, \quad \mathbf{\Lambda}_2\mathbf{\Lambda}_2 = \mathbf{\Lambda}_2, \quad \mathbf{\Lambda}_1\mathbf{\Lambda}_2 = 0. \tag{1.7}$$

We are now able to recall the optimal trace bounds for the EMT. For $d = 2, 3$, let

$$K_1 := \frac{1}{d(\tilde{\kappa} - \kappa)}\frac{d\tilde{\kappa} + 2(d-1)\mu}{d\kappa + 2(d-1)\mu}, \tag{1.8}$$

$$K_2 := \frac{1}{2(\tilde{\mu} - \mu)}\left[\frac{d^2 + d - 2}{2} + 2(\tilde{\mu} - \mu)\left(\frac{d-1}{2\mu} + \frac{d-1}{d\kappa + 2(d-1)\mu}\right)\right]. \tag{1.9}$$

The following trace bounds were obtained by Lipton [21] (see also [9]): Suppose $|\Omega| = 1$ and let $\mathbb{M}$ be the EMT associated with Ω, then we have

$$\operatorname{tr}\left(\mathbf{\Lambda}_1\mathbb{M}^{-1}\mathbf{\Lambda}_1\right) \leq K_1, \tag{1.10}$$

$$\operatorname{tr}\left(\mathbf{\Lambda}_2\mathbb{M}^{-1}\mathbf{\Lambda}_2\right) \leq K_2, \tag{1.11}$$

provided that $\tilde{\kappa} - \kappa > 0$. (If $\tilde{\kappa} - \kappa < 0$, the inequalities change the direction.) Since $\mathbf{\Lambda}_1\mathbb{M}^{-1}\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2\mathbb{M}^{-1}\mathbf{\Lambda}_2$ are block diagonal components for $\mathbb{M}^{-1}$, one can see that

$$\operatorname{tr}\mathbb{M}^{-1} = \operatorname{tr}\left(\mathbf{\Lambda}_1\mathbb{M}^{-1}\mathbf{\Lambda}_1\right) + \operatorname{tr}\left(\mathbf{\Lambda}_2\mathbb{M}^{-1}\mathbf{\Lambda}_2\right),$$

and hence
$$\operatorname{tr} \mathbb{M}^{-1} \leq K_1 + K_2. \tag{1.12}$$

Note that $\mathbf{\Lambda}_1 \mathbb{M} \mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2 \mathbb{M} \mathbf{\Lambda}_2$ are the bulk and shear parts of $\mathbb{M}$, respectively. We also note that (1.10) and (1.11) are lower bounds for $\mathbb{M}$ since they are upper bounds for $\mathbb{M}^{-1}$. It is worth emphasizing that upper bounds for $\mathbb{M}$ are also derived in [21]. In [9], it is shown that inclusions Ω whose trace is close to the upper bound must be infinitely thin. The upper and lower bounds for the EMT may also be derived as a low volume fraction limit of the Hashin-Shtrikman bounds for the effective moduli of the two phase composites, which was obtained by Zhikov [29, 30] and Milton-Kohn [24]. Benveniste [8] obtained the upper and lower bounds of EMTs when those EMTs happen to be isotropic. (See also [25].)

In this paper we are interested in the shape of the inclusion whose EMT satisfies the equality in either (1.10) or (1.11). This is an isoperimetric inequality for the EMT. In this direction we prove the following theorem:

Theorem 1.3 *Let Ω be a simply connected bounded domain in $\mathbb{R}^3$ with a Lipschitz boundary. Suppose $|\Omega| = 1$ and let $\mathbb{M}$ be the EMT associated with Ω. If the equality holds in either (1.10) or (1.11), then Ω is an ellipse in two dimensions and an ellipsoid in three dimensions.*

We remark that optimal shapes for a cavity (hole) in two dimension were investigated by Cherkaev *et al* [10] and Milton *et al* [26].

The dimension of the space of symmetric 4-tensors in the three dimensional space is 21, and hence the equalities (1.10) and (1.11) are satisfied on a 19 $(21 - 2)$ dimensional surface in tensor space. However ellipsoid geometries (with unit volume) only cover a 5 dimensional manifold within that 19 dimensional space.

It is interesting to notice similarity of Theorem 1.3 to the Pólya-Szegö conjecture, which asserts that the inclusion whose polarization tensor has the smallest trace is a disk or a ball. The Pólya-Szegö conjecture was proved to be true by Kang-Milton [18, 19]. As for the Pólya-Szegö conjecture, Theorem 1.3, which concerns elasticity, will be proved using Eshelby's conjecture.

In order to prove Theorem 1.3, we will show that if equality holds in (1.10), then the strain tensor corresponding to a certain uniform loading $\mathbf{A}$ (with a special structure) is constant in Ω, while if the equality holds in (1.11), then the strain tensors corresponding to five (two in 2D) linearly independent uniform loadings are constant in Ω. Thus in two dimensions the strong Eshelby conjecture immediately implies that the inclusion is an ellipse. However, in three dimension, the weak Eshelby conjecture does not guarantee that the inclusion is an ellipsoid. In order to apply the weak Eshelby conjecture, we need to have equalities in both (1.10) and (1.11), or the equality in the whole lower trace bound (1.12). But we are able to show additionally that if the equality holds in (1.10) then the eigenvalues of the strain tensor are all the same, and that if the equality holds in (1.11) then strains

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corresponding to five linearly independent loadings are constant. Thus thanks to Theorem 1.1 and Theorem 1.2 we are able to conclude that the inclusion is of ellipsoidal shape.

This paper is organized as follows. In section 2, we show that the displacement vectors can be decomposed in a way similar to the Helmholtz decomposition. This is done using the single layer potential for the Lamé system. Theorem 1.1 is proved in section 3, Theorem 1.2 in section 4, and Theorem 1.3 in section 4. The appendix is for the proof of Lemma 4.1 which is used to prove Theorem 1.2.

2 Single layer potential

Let us first recall the notion of the single layer potential for the Lamé operator $\mathcal{L}_{\mathbb{C}}\mathbf{u} := \nabla \cdot \mathbb{C}\mathcal{E}(\mathbf{u})$. The Kelvin matrix $\mathbf{\Gamma} = (\Gamma_{ij})_{i,j=1}^3$ of the fundamental solution to the Lamé operator $\mathcal{L}_{\mathbb{C}}$ in three dimensions is given by

$$\Gamma_{ij}(\mathbf{x}) := -\frac{\alpha_1}{4\pi}\frac{\delta_{ij}}{|\mathbf{x}|} - \frac{\alpha_2}{4\pi}\frac{x_i x_j}{|\mathbf{x}|^3}, \quad \mathbf{x} \neq 0, \tag{2.1}$$

where

$$\alpha_1 = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{2\mu + \lambda}\right) \quad \text{and} \quad \alpha_2 = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{2\mu + \lambda}\right). \tag{2.2}$$

The single layer potential of the vector valued density function $\mathbf{f}$ on $\partial\Omega$ associated with the Lamé parameters (λ, μ) is defined by

$$\mathcal{S}_\Omega[\mathbf{f}](\mathbf{x}) := \int_{\partial\Omega} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y})\mathbf{f}(\mathbf{y})\, d\sigma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{2.3}$$

Using the divergence theorem, we have

$$\mathcal{S}_\Omega[\mathbf{f}](\mathbf{x}) = -\frac{\alpha_1}{4\pi}\int_{\partial\Omega}\frac{\mathbf{f}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d\sigma(\mathbf{y}) - \frac{\alpha_2}{4\pi}\int_{\partial\Omega}\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}(\mathbf{x}-\mathbf{y})\cdot\mathbf{f}(\mathbf{y})d\sigma(\mathbf{y})$$

$$= -\frac{\alpha_1}{4\pi}\int_{\partial\Omega}\frac{\mathbf{f}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d\sigma(\mathbf{y}) + \frac{\alpha_2}{4\pi}\nabla\int_{\partial\Omega}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{f}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d\sigma(\mathbf{y})$$

$$\quad -\frac{\alpha_2}{4\pi}\int_{\partial\Omega}\frac{1}{|\mathbf{x}-\mathbf{y}|}\nabla_{\mathbf{x}}\left((\mathbf{x}-\mathbf{y})\cdot\mathbf{f}(\mathbf{y})\right)d\sigma(\mathbf{y})$$

$$= -\frac{\alpha_1+\alpha_2}{4\pi}\int_{\partial\Omega}\frac{\mathbf{f}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d\sigma(\mathbf{y}) + \frac{\alpha_2}{4\pi}\nabla\int_{\partial\Omega}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{f}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d\sigma(\mathbf{y})$$

$$= -\frac{\alpha_1+\alpha_2}{4\pi}\int_{\partial\Omega}\frac{\mathbf{f}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d\sigma(\mathbf{y}) + \frac{\alpha_2}{4\pi}\nabla\nabla\cdot\int_{\partial\Omega}|\mathbf{x}-\mathbf{y}|\mathbf{f}(\mathbf{y})d\sigma(\mathbf{y}).$$

Since $\Delta|\mathbf{x}| = 2|\mathbf{x}|^{-1}$, we have

$$\mathcal{S}_\Omega[\mathbf{f}](\mathbf{x}) = -\frac{\alpha_1+\alpha_2}{8\pi}\Delta\int_{\partial\Omega}|\mathbf{x}-\mathbf{y}|\mathbf{f}(\mathbf{y})d\sigma(\mathbf{y}) + \frac{\alpha_2}{4\pi}\nabla\nabla\cdot\int_{\partial\Omega}|\mathbf{x}-\mathbf{y}|\mathbf{f}(\mathbf{y})d\sigma(\mathbf{y}). \tag{2.4}$$

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Let

$$\mathcal{H}_\Omega[\mathbf{f}](\mathbf{x}) := \frac{1}{4\pi} \int_{\partial\Omega} |\mathbf{x} - \mathbf{y}| \mathbf{f}(\mathbf{y}) d\sigma(\mathbf{y}) \ . \tag{2.5}$$

Then, in summary, we have

$$\mathcal{S}_\Omega[\mathbf{f}](\mathbf{x}) = -\frac{\alpha_1 + \alpha_2}{2} \Delta \mathcal{H}_\Omega[\mathbf{f}](\mathbf{x}) + \alpha_2 \nabla\nabla \cdot \mathcal{H}_\Omega[\mathbf{f}](\mathbf{x}) \ . \tag{2.6}$$

It is worth emphasizing that $\Delta^2 \mathcal{H}_\Omega[\mathbf{f}] = 0$, *i.e.*, $\mathcal{H}_\Omega[\mathbf{f}]$ is biharmonic, in Ω and $\mathbb{R}^3 \backslash \overline{\Omega}$. Thus (2.6) shows that the solution to the Lamé system in a bounded domain in Ω or the exterior $\mathbb{R}^3 \setminus \overline{\Omega}$ can be decomposed into a part harmonic in Ω or $\mathbb{R}^3 \setminus \overline{\Omega}$ and a gradient part.

Suppose that the solution $\mathbf{u}$ to (1.4) inside Ω is given by

$$\mathbf{u}(\mathbf{x}) = \mathbf{B}\mathbf{x} + \mathbf{v}, \quad \mathbf{x} \in \Omega \tag{2.7}$$

for some constant symmetric matrix $\mathbf{B}$ and a constant vector $\mathbf{v}$. Then the solution is given by

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \mathbf{A}\mathbf{x} + \mathcal{S}_\Omega[\mathbf{f}](\mathbf{x}) \ , & \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega} \ , \\ \mathbf{B}\mathbf{x} + \mathbf{v} \ , & \mathbf{x} \in \Omega \ , \end{cases} \tag{2.8}$$

where

$$\mathbf{f} = (\mathbb{C}_1 - \mathbb{C}_0)\mathcal{E}(\mathbf{B}\mathbf{x})\mathbf{n} \ . \tag{2.9}$$

Here $\mathbf{n} = (n_1, n_2, n_3)$ is the unit outward normal vector field to $\partial\Omega$. See [19, Section 4]. Note that

$$(\mathbb{C}_1 - \mathbb{C}_0)\mathcal{E}(\mathbf{B}\mathbf{x})\mathbf{n} = [(\tilde{\lambda} - \lambda)\operatorname{tr}(\mathbf{B})\mathbf{I} + 2(\tilde{\mu} - \mu)\mathbf{B}]\mathbf{n}. \tag{2.10}$$

Let us put

$$\mathbf{B}^* := (\tilde{\lambda} - \lambda)\operatorname{tr}(\mathbf{B})\mathbf{I} + 2(\tilde{\mu} - \mu)\mathbf{B}, \tag{2.11}$$

so that $\mathbf{f}$ in (2.8) is given by

$$\mathbf{f} = \mathbf{B}^*\mathbf{n}. \tag{2.12}$$

According to (2.4), we have

$$\mathcal{S}_\Omega[\mathbf{B}^*\mathbf{n}](\mathbf{x}) = -\frac{\alpha_1 + \alpha_2}{2} \Delta \mathcal{H}_\Omega[\mathbf{B}^*\mathbf{n}](\mathbf{x}) + \alpha_2 \nabla\nabla \cdot \mathcal{H}_\Omega[\mathbf{B}^*\mathbf{n}](\mathbf{x}) \ . \tag{2.13}$$

One can easily see that

$$\mathcal{H}_\Omega[\mathbf{B}^*\mathbf{n}](\mathbf{x}) = \mathbf{B}^*\mathcal{H}_\Omega[\mathbf{n}](\mathbf{x}) = -\mathbf{B}^*\nabla p_\Omega(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2.14}$$

where p_Ω is defined by

$$p_\Omega(\mathbf{x}) := \frac{1}{4\pi} \int_\Omega |\mathbf{x} - \mathbf{y}| d\mathbf{y} \ , \quad \mathbf{x} \in \mathbb{R}^3 \ . \tag{2.15}$$

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Therefore we have

$$\mathcal{S}_\Omega[\mathbf{B}^*\mathbf{n}](\mathbf{x}) = \frac{\alpha_1 + \alpha_2}{2}\mathbf{B}^*\nabla\Delta p_\Omega(\mathbf{x}) - \alpha_2\nabla\nabla\cdot\mathbf{B}^*\nabla p_\Omega(\mathbf{x}).$$

For a 3×3 symmetric matrix $\mathbf{B}$, let $\Delta_\mathbf{B} := \nabla\cdot\mathbf{B}\nabla$. We note that $\Delta_\mathbf{I} = \Delta$, the usual Laplacian. We then define

$$w_\Omega^\mathbf{B}(\mathbf{x}) := \Delta_\mathbf{B}p_\Omega(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{2.16}$$

In particular, we write $w_\Omega = w_\Omega^\mathbf{I}$. Then, one can easily see that

$$w_\Omega(\mathbf{x}) := \frac{2}{4\pi}\int_\Omega\frac{1}{|\mathbf{x} - \mathbf{y}|}d\sigma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3, \tag{2.17}$$

which is (2 times) the Newtonian potential of Ω.

It is appropriate to recall now the proof of the weak Eshelby conjecture by Kang and Milton. In [19], the matter was reduced to the statement: 'The Newtonian potential is quadratic in Ω if and only if Ω is an ellipsoid', which was proved by Dive [12] and Nikliborc [27] in relation to the Newtonian potential problem (see also [11]). This statement can be rephrased as

$$w_\Omega \text{ is quadratic in } \Omega \text{ if and only if } \Omega \text{ is an ellipsoid,} \tag{2.18}$$

If we further put $\alpha := \frac{\alpha_1+\alpha_2}{2\alpha_2}$, then we have

$$\mathcal{S}_\Omega[\mathbf{B}^*\mathbf{n}](\mathbf{x}) = \frac{1}{\alpha_2}\left[\alpha\mathbf{B}^*\nabla w_\Omega(\mathbf{x}) - \nabla w_\Omega^{\mathbf{B}^*}(\mathbf{x})\right]. \tag{2.19}$$

We emphasize that $\alpha > 1$.

3 Proof of Theorem 1.1

Suppose that the solution $\mathbf{u}$ to (1.4) is linear in Ω and given by (2.8). Then by (2.12) we have

$$\mathcal{S}_\Omega[\mathbf{B}^*\mathbf{n}](\mathbf{x}) = (\mathbf{B} - \mathbf{A})\mathbf{x} + \mathbf{v}, \quad \mathbf{x} \in \Omega.$$

It then follows from (2.19) that

$$\alpha\mathbf{B}^*\nabla w_\Omega(\mathbf{x}) - \nabla w_\Omega^{\mathbf{B}^*}(\mathbf{x}) = \alpha_2(\mathbf{B} - \mathbf{A})\mathbf{x} + \alpha_2\mathbf{v}, \quad \mathbf{x} \in \Omega. \tag{3.1}$$

Note that if eigenvalues of $\mathbf{B}$ are either all the same or all distinct, so are those of $\mathbf{B}^*$. After rotation if necessary, we may assume that $\mathbf{B}^*$ is diagonal, say

$$\mathbf{B}^* = \text{diag}[b_1, b_2, b_3]. \tag{3.2}$$

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(i) Suppose first that all eigenvalues of $\mathbf{B}^*$ are the same, *i.e.*, $b_1 = b_2 = b_3 = b$. In this case, since $w_\Omega^{\mathbf{B}^*} = bw_\Omega$, it follows from (3.1) that

$$b(\alpha - 1)\nabla w_\Omega = \text{linear in } \Omega.$$

Since $\alpha > 1$, w_Ω is quadratic in Ω, and hence Ω is an ellipsoid by (2.18).

(ii) Suppose now that all eigenvalues of $\mathbf{B}^*$ are distinct, *i.e.*, $b_i \neq b_j$ if $i \neq j$. In this case, (3.1) yields that

$$\frac{\partial}{\partial x_j}\left(\alpha b_j w_\Omega - w_\Omega^{\mathbf{B}^*}\right) = \text{linear in } \Omega, \quad j = 1, 2, 3,$$

and hence

$$\alpha b_j w_\Omega - w_\Omega^{\mathbf{B}^*} \approx f_j(\mathbf{x}) \quad \text{in } \Omega, \quad j = 1, 2, 3,$$

for some function f_j which is independent of x_j. Here and afterwards $\approx$ denotes the equality up to a quadratic function. It then follows that

$$\alpha w_\Omega \approx \frac{f_1 - f_2}{b_1 - b_2}, \quad w_\Omega^{\mathbf{B}^*} \approx \frac{b_2 f_1 - b_1 f_2}{b_1 - b_2}, \tag{3.3}$$

and

$$(b_3 - b_2)f_1 + (b_1 - b_3)f_2 + (b_2 - b_1)f_3 \approx 0. \tag{3.4}$$

Since f_j is independent of x_j for $j = 1, 2, 3$, one can easily see that (3.4) holds only when f_1, f_2 and f_3 take the form

$$f_1(\mathbf{x}) \approx \frac{m(x_3) - n(x_2)}{b_3 - b_2},$$
$$f_2(\mathbf{x}) \approx \frac{r(x_1) - m(x_3)}{b_1 - b_3},$$
$$f_3(\mathbf{x}) \approx \frac{n(x_2) - r(x_1)}{b_2 - b_1},$$

for some functions m, n and r. It then follows from (3.3) that

$$\alpha w_\Omega \approx \frac{m(x_3)}{(b_3 - b_2)(b_1 - b_3)} + \frac{n(x_2)}{(b_2 - b_1)(b_3 - b_2)} + \frac{r(x_1)}{(b_1 - b_3)(b_2 - b_1)}. \tag{3.5}$$

Since $\Delta w_\Omega = 2$ in Ω, we have

$$\frac{m''(x_3)}{(b_3 - b_2)(b_1 - b_3)} + \frac{n''(x_2)}{(b_2 - b_1)(b_3 - b_2)} + \frac{r''(x_1)}{(b_1 - b_3)(b_2 - b_1)} = \text{constant}.$$

Thus r, n, and m are quadratic functions of x_1, x_2, and x_3, respectively, and hence w_Ω is quadratic in Ω. Thus Ω is an ellipsoid.

This completes the proof. $\square$

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4 Proof of Theorem 1.2

In order to prove Theorem 1.2, we use the following lemma whose proof will be given in the appendix.

Lemma 4.1 *Let* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *be two symmetric* 3×3*-matrices. If* $\mathbf{B}_1 + t\mathbf{B}_2$ *has a multiple eigenvalue for all real numbers* t*, then* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *can be diagonalized by the same orthogonal matrix.*

Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two linearly independent symmetric 3×3 matrices and suppose that solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ to (1.4) with $\mathbf{A} = \mathbf{A}_1$ and $\mathbf{A} = \mathbf{A}_2$ are linear in Ω. Put $\mathbf{B}_j := \mathcal{E}(\mathbf{u}_j)$, $j = 1, 2$. Since the EMT is positive or negative definite on M_d^s (Theorem 10.6 of [5]), (1.6) shows that $\mathbf{B}_1$ and $\mathbf{B}_2$ are linearly independent. According to (2.12) we have

$$\mathcal{S}_\Omega[\mathbf{B}_1^*\mathbf{n}](\mathbf{x}) = (\mathbf{B}_1 - \mathbf{A}_1)\mathbf{x} + \mathbf{v}_1, \quad \mathbf{x} \in \Omega,$$
$$\mathcal{S}_\Omega[\mathbf{B}_2^*\mathbf{n}](\mathbf{x}) = (\mathbf{B}_2 - \mathbf{A}_2)\mathbf{x} + \mathbf{v}_2, \quad \mathbf{x} \in \Omega.$$

It then follows from (2.19) that

$$\begin{cases} \alpha\mathbf{B}_1^*\nabla w_\Omega(\mathbf{x}) - \nabla w_\Omega^{\mathbf{B}_1^*}(\mathbf{x}) = \alpha_2(\mathbf{B}_1 - \mathbf{A}_1)\mathbf{x} + \alpha_2\mathbf{v}_1, \\ \alpha\mathbf{B}_2^*\nabla w_\Omega(\mathbf{x}) - \nabla w_\Omega^{\mathbf{B}_2^*}(\mathbf{x}) = \alpha_2(\mathbf{B}_2 - \mathbf{A}_2)\mathbf{x} + \alpha_2\mathbf{v}_2, \end{cases} \tag{4.1}$$

for $\mathbf{x} \in \Omega$.

Let us suppose that all of $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_1 + t\mathbf{B}_2$ $(t \in \mathbb{R})$ have an eigenvalue of multiplicity 2 (otherwise we apply Theorem 1.1 to conclude that Ω is an ellipsoid). By Lemma 4.1, $\mathbf{B}_1$ and $\mathbf{B}_2$ can be diagonalized by a single orthogonal matrix. Thus we may assume that $\mathbf{B}_1$ and $\mathbf{B}_2$ are diagonal. Then from (2.11) $\mathbf{B}_1^*$ and $\mathbf{B}_2^*$ are also diagonal and we may let

$$\mathbf{B}_1^* = \text{diag}[b_1, b_1, c_1], \quad \mathbf{B}_2^* = \text{diag}[b_2, b_2, c_2],$$

where $b_1 \neq c_1$ and $b_2 \neq c_2$. Since $\mathbf{B}_1^*$ and $\mathbf{B}_2^*$ are linearly independent, we have

$$b_1 c_2 \neq c_1 b_2.$$

By (4.1), we have

$$\alpha b_1 w_\Omega - w_\Omega^{\mathbf{B}_1^*} \approx f(x_3), \tag{4.2}$$
$$\alpha c_1 w_\Omega - w_\Omega^{\mathbf{B}_1^*} \approx g(x_1, x_2), \tag{4.3}$$
$$\alpha b_2 w_\Omega - w_\Omega^{\mathbf{B}_2^*} \approx h(x_3), \tag{4.4}$$
$$\alpha c_2 w_\Omega - w_\Omega^{\mathbf{B}_2^*} \approx l(x_1, x_2), \tag{4.5}$$

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for some functions f, g, h, and l. Here again $\approx$ denotes the equality up to a quadratic function. By (2.16), we have from (4.2) and (4.4)

$$(\alpha - 1)b_2 f(x_3) - (\alpha - 1)b_1 h(x_3) \approx (\alpha - 1)(b_1 c_2 - b_2 c_1)\frac{\partial^2 p_\Omega}{\partial x_3^2}, \qquad (4.6)$$

and from (4.3) and (4.5)

$$(b_2 - \alpha c_2)g(x_1, x_2) - (b_1 - \alpha c_1)l(x_1, x_2) \approx (1 - \alpha)(b_1 c_2 - b_2 c_1)\frac{\partial^2 p_\Omega}{\partial x_3^2}. \qquad (4.7)$$

It then follows that

$$\frac{\partial^2 p_\Omega}{\partial x_3^2} \approx 0. \qquad (4.8)$$

We then obtain from (4.2)-(4.5) that

$$(\alpha - 1)b_1 w_\Omega \approx f(x_3), \quad (\alpha - 1)b_2 w_\Omega \approx h(x_3),$$

and

$$(\alpha c_1 - b_1)w_\Omega \approx g(x_1, x_2), \quad (\alpha c_2 - b_2)w_\Omega \approx l(x_1, x_2).$$

Thus we conclude that $w_\Omega \approx 0$, and hence Ω is an ellipsoid.

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

5   Proof of Theorem 1.3

The space M_d^s is equipped with the inner product $\mathbf{A} : \mathbf{B}$, where $\mathbf{A} : \mathbf{B}$ denotes the contraction of two matrices $\mathbf{A}$ and $\mathbf{B}$, *i.e.*, $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij}b_{ij} = \operatorname{tr}(\mathbf{A}^T\mathbf{B})$ where $\operatorname{tr}(\mathbf{A})$ denotes the trace of $\mathbf{A}$. For $d = 2, 3$, let $d_* := \frac{d(d+1)}{2}$, which is the dimension of M_d^s. Let $\mathbf{B}_1 = \frac{1}{\sqrt{d}}\mathbf{I}_2$ be a basis for $\mathbf{\Lambda}_1(M_d^s)$ (of a unit length), and $\{\mathbf{B}_2, \ldots, \mathbf{B}_{d_*}\}$ be an orthonormal basis for $\mathbf{\Lambda}_2(M_d^s)$. Then $\{\mathbf{B}_1, \ldots, \mathbf{B}_{d_*}\}$ is an orthonormal basis for M_d^s, *i.e.*,

$$\mathbf{B}_i : \mathbf{B}_j = \delta_{ij},$$

where δ_{ij} is Kronecker's delta. Note that for any symmetric 4-tensor $\mathbb{T}$, we have

$$\operatorname{tr}\mathbb{T} = \sum_{k=1}^{d_*}\mathbb{T}\mathbf{B}_k : \mathbf{B}_k. \qquad (5.1)$$

We deal with the case when $\mathbb{C}^1 - \mathbb{C}^0$ is positive definite so that $\mathbb{M}$ is a symmetric positive-definite linear operator on M_d^s. The other case can be treated in the exactly same way.

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Let us first invoke some facts proved in [9]. Introduce a 4-tensor $\widetilde{\mathbb{M}}$ by

$$
\begin{aligned}
\mathbf{A} : \widetilde{\mathbb{M}}\mathbf{A} \quad = \quad & \min_{\mathbf{v} \in H^1\left(\mathbb{R}^d\right)} \int_{\mathbb{R}^d} \mathbb{C}_\Omega \left(\mathcal{E}(\mathbf{v}) + 1_\Omega \mathbb{G}\mathbf{A}\right) : \left(\mathcal{E}(\mathbf{v}) + 1_\Omega \mathbb{G}\mathbf{A}\right) dx \\
& + |\Omega| \left(\mathbb{C}^1 - \mathbb{C}^0\right) \left(\mathbb{C}^1\right)^{-1} \mathbb{C}^0 \mathbf{A} : \mathbf{A}
\end{aligned}
\tag{5.2}
$$

for $\mathbf{A} \in M_d^s$, where

$$
\mathbb{G} := \mathbb{I} - (\mathbb{C}^1)^{-1}\mathbb{C}^0.
\tag{5.3}
$$

Note that the minimum in (5.2) is attained by $\mathbf{v} = \mathbf{u} - \mathbf{A}\mathbf{x}$, where $\mathbf{u}$ is the solution of (1.4). It is proved in [9, Corollary 3.2] that

$$
\boldsymbol{\Lambda}_1 \widetilde{\mathbb{M}} \boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}_1 \mathbb{M} \boldsymbol{\Lambda}_1 \quad \text{and} \quad \boldsymbol{\Lambda}_2 \widetilde{\mathbb{M}} \boldsymbol{\Lambda}_2 = \boldsymbol{\Lambda}_2 \mathbb{M} \boldsymbol{\Lambda}_2 \,.
\tag{5.4}
$$

In particular, we have

$$
\text{tr} \left(\boldsymbol{\Lambda}_1 \widetilde{\mathbb{M}}^{-1} \boldsymbol{\Lambda}_1\right) = \text{tr} \left(\boldsymbol{\Lambda}_1 \mathbb{M}^{-1} \boldsymbol{\Lambda}_1\right) \quad \text{and} \quad \text{tr} \left(\boldsymbol{\Lambda}_2 \widetilde{\mathbb{M}}^{-1} \boldsymbol{\Lambda}_2\right) = \text{tr} \left(\boldsymbol{\Lambda}_2 \mathbb{M}^{-1} \boldsymbol{\Lambda}_2\right).
\tag{5.5}
$$

Let $\mathbb{C}$ be an isotropic 4-tensor, $i.e.$,

$$
\mathbb{C} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbb{I} = d\kappa \boldsymbol{\Lambda_1} + 2\mu \boldsymbol{\Lambda_2},
$$

for some λ and μ satisfying (1.3). Let $L^2\left(\mathbb{R}^d, M_d^s\right)$ be the space of square integrable functions on $\mathbb{R}^d$ valued in M_d^s and $H^1(\mathbb{R}^d, M_d^s)$ the Sobolev space. For $\mathbf{P} \in L^2\left(\mathbb{R}^d, M_d^s\right)$, we define $F_{\mathbb{C}}(\mathbf{P})$ by

$$
F_{\mathbb{C}}(\mathbf{P}) := -\mathcal{E}\mathcal{L}_{\mathbb{C}}^{-1}(\nabla \cdot \mathbf{P}),
\tag{5.6}
$$

where $\mathcal{L}_{\mathbb{C}} = \nabla \cdot \mathbb{C}\mathcal{E}$. In other words, if Φ is a unique solution in $H^1(\mathbb{R}^d, M_d^s)$ to

$$
\mathcal{L}_{\mathbb{C}}\left(\Phi\right) + \nabla \cdot \mathbf{P} = 0,
\tag{5.7}
$$

then $F_{\mathbb{C}}(\mathbf{P})$ is given by

$$
F_{\mathbb{C}}(\mathbf{P}) = \mathcal{E}\left(\Phi\right).
$$

If Φ is the solution to (5.7), then

$$
\int_{\mathbb{R}^d} \mathbb{C}\left(\mathcal{E}\left(\Phi\right) + \mathbb{C}^{-1}\mathbf{P}\right) : \mathcal{E}\left(\Psi\right) = 0
$$

for all $\Psi \in H^1(\mathbb{R}^d, M_d^s)$, and hence by taking $\Psi = \Phi$ we have

$$
\int_{\mathbb{R}^d} \mathbf{P} : F_{\mathbb{C}}(\mathbf{P}) = -\int_{\mathbb{R}^d} \mathbb{C}F_{\mathbb{C}}(\mathbf{P}) : F_{\mathbb{C}}(\mathbf{P})
\tag{5.8}
$$

We prove Theorem 1.3 using the following two propositions whose proofs will be given at the end of this section.

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Proposition 5.1 *Let*

$$E_{\mathbf{A}}(\mathbb{C}^0, \mathbf{P}) = \int_{\Omega} \mathbf{P} : F_{\mathbb{C}^0}(1_{\Omega}\mathbf{P}) + \int_{\Omega} \left(\mathbb{C}^0 - \mathbb{C}^1\right)^{-1} \mathbf{P} : \mathbf{P} + 2\int_{\Omega} \mathbf{P} : \mathbf{A}. \qquad (5.9)$$

Then the following holds

$$\mathbf{A} : \widetilde{\mathbb{M}}\mathbf{A} = \sup_{\mathbf{P} \in L^2(\mathbb{R}^d : M_d^s)} E_{\mathbf{A}}(\mathbb{C}^0, \mathbf{P}). \qquad (5.10)$$

Furthermore, this supremum is attained by $\mathbf{P} = 1_{\Omega}\left(\mathbb{C}^1 - \mathbb{C}^0\right)\mathcal{E}(\mathbf{u})$, *where* $\mathbf{u}$ *is the solution of (1.4).*

We then show that structures reaching the lower trace bounds have a particular structure, as explained by the proposition below.

Proposition 5.2 *If equality in (1.10) holds, then we have*

$$E_{\mathbf{A}}(\mathbb{C}^0, 1_{\Omega}\mathbf{B}_1) = \sup_{\mathbf{P} \in L^2(\mathbb{R}^d : M_d^s)} E_{\mathbf{A}}(\mathbb{C}^0, \mathbf{P}) \qquad (5.11)$$

with $\mathbf{A} = \mathbb{M}^{-1}\mathbf{B}_1$. *If equality in (1.11) holds, then we have*

$$E_{\mathbf{A}}(\mathbb{C}^0, 1_{\Omega}\mathbf{B}_k) = \sup_{\mathbf{P} \in L^2(\mathbb{R}^d : M_d^s)} E_{\mathbf{A}}(\mathbb{C}^0, \mathbf{P}) \qquad (5.12)$$

with $\mathbf{A} = \mathbb{M}^{-1}\mathbf{B}_k$, *for* $k = 2, \ldots, d_*$.

Proof of Theorem 1.3. Introduce a bilinear form $\mathcal{F}_{\mathbb{C}^0}(\mathbf{Q}, \mathbf{R})$ by

$$\mathcal{F}_{\mathbb{C}^0}(\mathbf{Q}, \mathbf{R}) = \int_{\Omega} \mathbf{Q} : F_{\mathbb{C}^0}(1_{\Omega}\mathbf{R}) + \int_{\Omega} \left(\mathbb{C}^0 - \mathbb{C}^1\right)^{-1} \mathbf{Q} : \mathbf{R}.$$

It follows from (5.8) that

$$\mathcal{F}_{\mathbb{C}^0}(1_{\Omega}\mathbf{Q}, 1_{\Omega}\mathbf{Q}) = -\int_{\Omega} \mathbb{C}^0 F_{\mathbb{C}^0}(1_{\Omega}\mathbf{Q}) : F_{\mathbb{C}^0}(1_{\Omega}\mathbf{Q}) - \int_{\Omega} \left(\mathbb{C}^1 - \mathbb{C}^0\right)^{-1}(1_{\Omega}\mathbf{Q}) : (1_{\Omega}\mathbf{Q})$$

$$\leq -\int_{\Omega} \left(\mathbb{C}^1 - \mathbb{C}^0\right)^{-1}(1_{\Omega}\mathbf{Q}) : (1_{\Omega}\mathbf{Q})$$

$$\leq -K \left\|1_{\Omega}\mathbf{Q}\right\|_{L^2(\mathbb{R}^d : M_d^s)}$$

for some positive constant K. The last holds due to the positive-definiteness of $\mathbb{C}^1 - \mathbb{C}^0$. As a consequence, $\mathcal{F}_{\mathbb{C}^0}$ is negative definite when restricted to $H = \{\mathbf{P} \in L^2(\Omega : M_d^s)^2$, supported in $\Omega\}$. Therefore $E_{\mathbf{A}}(\mathbb{C}^0, \mathbf{Q}) = \mathcal{F}_{\mathbb{C}^0}(\mathbf{Q}, \mathbf{Q}) + 2\int_{\Omega} \mathbf{Q} : \mathbf{A}$ is a strictly concave functional on H, and therefore admits at most one maximizer in H.

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We observe that since $\mathbb{C}^1 - \mathbb{C}^0$ is isotropic, if $\mathbf{B}$ is diagonal and all the eigenvalues are the same, so is $(\mathbb{C}^1 - \mathbb{C}^0)^{-1}\mathbf{B}$. If $\mathbf{B}$ is trace-free and all the eigenvalues are distinct, so is $(\mathbb{C}^1 - \mathbb{C}^0)^{-1}\mathbf{B}$.

Suppose that equality holds in (1.10). It then follows from Proposition 5.1, (5.2), and uniqueness of the maximizer in Ω that

$$\left(\mathbb{C}^1 - \mathbb{C}^0\right)\mathcal{E}(\mathbf{u}_1) = \mathbf{B}_1 \text{ in } \Omega,$$

where $\mathbf{u}_1$ is the solution to (1.4) with $\mathbf{A} = \mathbb{M}^{-1}\mathbf{B}_1$. Recall that $\mathbf{B}_1 = \frac{1}{\sqrt{d}}\mathbf{I}$. Therefore, $\mathcal{E}(\mathbf{u}_1)$ is constant in Ω and all the eigenvalues of $\mathcal{E}(\mathbf{u}_1)$ are the same. Thus Ω is an ellipse or an ellipsoid due to Theorem 1.1.

Suppose now that equality holds in (1.11). Then for similar reasons we can deduce that for each $k = 2, \ldots, d_*$,

$$\left(\mathbb{C}^1 - \mathbb{C}^0\right)\mathcal{E}(\mathbf{u}_k) = \mathbf{B}_k \text{ in } \Omega,$$

where $\mathbf{u}_k$ is the solution to (1.4) with $\mathbf{A} = \mathbb{M}^{-1}\mathbf{B}_k$. Thus Ω is an ellipse or an ellipsoid due to Theorem 1.2.

This completes the proof. $\qquad\square$

Proof of Proposition 5.1. Following the notation of [9], we define $W_{\mathbf{A}}(\mathbb{C}, \mathbf{P})$, for $\mathbf{A} \in M_d^s$, by

$$W_{\mathbf{A}}(\mathbb{C}, \mathbf{P}) = \int_{\mathbb{R}^d} \mathbf{P} : F_{\mathbb{C}}\mathbf{P} + \int_{\mathbb{R}^d} (\mathbb{C} - \mathbb{C}_\Omega)^{-1} \mathbf{P} : \mathbf{P} + 2\int_\Omega \mathbf{P} : \left(\mathbb{C}^1 - \mathbb{C}\right)^{-1}\left(\mathbb{C}^1 - \mathbb{C}^0\right)\mathbf{A}.$$

It is proved in [9, Proposition 4.1], following the variational strategy given in [20] for the derivation of Hashin-Shtrikman type bounds, that for any isotropic elasticity tensor $\mathbb{C} < \mathbb{C}^0(< \mathbb{C}^1)$ we have

$$\mathbf{A} : \widetilde{\mathbb{M}}\mathbf{A} = \mathbf{A} : \left(\mathbb{C}^1 - \mathbb{C}^0\right)\left(\mathbb{C} - \mathbb{C}^1\right)^{-1}\left(\mathbb{C} - \mathbb{C}^0\right)\mathbf{A} + \sup_{\mathbf{P} \in L^2\left(\mathbb{R}^d : M_d^s\right)} W_{\mathbf{A}}(\mathbb{C}, \mathbf{P}). \quad (5.13)$$

Note that the supremum is attained by

$$\mathbf{P} = 1_\Omega \left(\mathbb{C}^1 - \mathbb{C}^0\right)\mathbf{A} + (\mathbb{C}_\Omega - \mathbb{C})(\mathcal{E}(\mathbf{u}) - \mathbf{A}), \quad (5.14)$$

where $\mathbf{u}$ is the solution to (1.4) with $\mathbf{A}$ in above identity. Since $(\mathbb{C}^1 - \mathbb{C}^0)(\mathbb{C} - \mathbb{C}^1)^{-1}(\mathbb{C} - \mathbb{C}^0)$ is positive definite, by sending $\mathbb{C}$ to $\mathbb{C}^0$, and restricting the supremum to fields $\mathbf{P}$ such that $1_\Omega\mathbf{P} = \mathbf{P}$ we obtain

$$\mathbf{A} : \widetilde{\mathbb{M}}\mathbf{A} \geq \sup_{\mathbf{P} \in L^2(\mathbb{R}^d, M_d^s)} E_{\mathbf{A}}(\mathbb{C}^0, \mathbf{P}). \quad (5.15)$$

For any $\mathbf{P} \in L^2\left(\mathbb{R}^d, M_d^s\right)$, and any positive definite isotropic elasticity tensor $\mathbb{C} < \mathbb{C}^0$, we define $E_{\mathbf{A}}(\mathbb{C}, \mathbf{P})$, for $\mathbf{A} \in M_d^s$, by

$$E_{\mathbf{A}}(\mathbb{C}, \mathbf{P}) = \int_\Omega \mathbf{P} : F_{\mathbb{C}}\left(1_\Omega\mathbf{P}\right) + \int_\Omega (\mathbb{C} - \mathbb{C}^1)^{-1}\mathbf{P} : \mathbf{P} + 2\int_\Omega \mathbf{P} : \left(\mathbb{C}^1 - \mathbb{C}\right)^{-1}\left(\mathbb{C}^1 - \mathbb{C}^0\right)\mathbf{A}.$$

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Note that this definition is consistent of that of $E_{\mathbf{A}}(\mathbb{C}^0, \mathbf{P})$ given in (5.9) by passing to the limit in $\mathbb{C}$.

Introducing the decomposition $\mathbf{P} = \mathbf{P}_\Omega + \mathbf{P}_U$, with $\mathbf{P}_\Omega 1_\Omega = \mathbf{P}_\Omega$, and $\mathbf{P}_U 1_\Omega \equiv 0$, we have

$$
\begin{aligned}
W_{\mathbf{A}}(\mathbb{C}, \mathbf{P}) &= E_{\mathbf{A}}(\mathbb{C}, \mathbf{P}_\Omega) + W_{\mathbf{A}}(\mathbb{C}, \mathbf{P}_U) \\
&+ \int_{\mathbb{R}^d} \mathbf{P}_\Omega : F_{\mathbb{C}} \mathbf{P}_U + \int_{\mathbb{R}^d} \mathbf{P}_U : F_{\mathbb{C}} \mathbf{P}_\Omega.
\end{aligned}
$$

Let $\mathbb{C} = \mathbb{C}_0 - \varepsilon \mathbb{I}$, where $\varepsilon > 0$. Then we have

$$
W_{\mathbf{A}}(\mathbb{C}, \mathbf{P}) = E_{\mathbf{A}}(\mathbb{C}, \mathbf{P}_\Omega) - \epsilon^{-1} \|\mathbf{P}_U\|^2_{L^2(\mathbb{R}^d)} + R(\mathbf{P}_U, \mathbf{P}_\Omega),
$$

where

$$
R(\mathbf{P}_U, \mathbf{P}_\Omega) := \int_{\mathbb{R}^d} \mathbf{P}_\Omega : F_{\mathbb{C}} \mathbf{P}_U + \int_{\mathbb{R}^d} \mathbf{P}_U : F_{\mathbb{C}} \mathbf{P}_\Omega + \int_{\mathbb{R}^d} \mathbf{P}_U : F_{\mathbb{C}} \mathbf{P}_U.
$$

By integration by parts, and by the Cauchy-Schwartz inequality, we readily obtain that for ε small enough $R(\mathbf{P}_U, \mathbf{P}_\Omega)$ satisfies

$$
|R(\mathbf{P}_U, \mathbf{P}_\Omega)| \leq K \|\mathbf{P}_U\|_{L^2(\mathbb{R}^d)} \left(\|\mathbf{P}_U\|_{L^2(\mathbb{R}^d)} + \|\mathbf{P}_\Omega\|_{L^2(\mathbb{R}^d)} \right),
$$

where the constant K is independent of $\mathbf{P}_U$, $\mathbf{P}_\Omega$, and ε. As a consequence, for ε small enough,

$$
-\epsilon^{-1} \|\mathbf{P}_U\|^2_{L^2(\mathbb{R}^d)} + R(\mathbf{P}_U, \mathbf{P}_\Omega) \leq 3K\varepsilon \|\mathbf{P}_\Omega\|^2_{L^2(\mathbb{R}^d)}.
$$

Note that from (5.8) $\int_\Omega \mathbf{P}_\Omega : F_{\mathbb{C}}(\mathbf{P}_\Omega)$ is negative definite, therefore

$$
E_{\mathbf{A}}(\mathbb{C}, \mathbf{P}) \leq \tilde{K} \|\mathbf{P}_\Omega\|_{L^2(\mathbb{R}^d)} (-\|\mathbf{P}_\Omega\|_{L^2(\mathbb{R}^d)} + 1),
$$

where $\tilde{K}$ is another constant independent of $\mathbf{P}_\Omega$ and ε. Thus $\|\mathbf{P}_\Omega\|_{L^2(\mathbb{R}^d)}$ must stay bounded, uniformly with respect to ϵ, close to the supremum. Taking the limit as ε tends to zero we obtain (5.10). Replacing $\mathbb{C}$ by $\mathbb{C}^0$ in (5.14) concludes the proof. $\square$

Proof of Proposition 5.2. Given $k \in \{1, \ldots, d_*\}$, choose $\mathbf{A} = \mathbb{M}^{-1} \mathbf{\Lambda}_l(\mathbf{B}_k)$ and use a test function $\mathbf{P} = 1_\Omega \mathbf{\Lambda}_l(\mathbf{B}_k)$ in (5.10). This gives

$$
\begin{aligned}
\mathbb{M}^{-1} \mathbf{\Lambda}_l(\mathbf{B}_k) : \mathbf{\Lambda}_l(\mathbf{B}_k) &\geq W_{\mathbf{A}}(\mathbb{C}^0, 1_\Omega \mathbf{\Lambda}_l(\mathbf{B}_k)) \qquad\qquad (5.16) \\
&= \int_\Omega \mathbf{\Lambda}_l(\mathbf{B}_k) : F_{\mathbb{C}^0}(1_\Omega \mathbf{\Lambda}_l(\mathbf{B}_k)) + \int_\Omega \left(\mathbb{C}^0 - \mathbb{C}^1 \right)^{-1} \mathbf{\Lambda}_l(\mathbf{B}_k) : \mathbf{\Lambda}_l(\mathbf{B}_k) \\
&+ 2 \int_\Omega \mathbf{\Lambda}_l(\mathbf{B}_k) : \mathbb{M}^{-1} \mathbf{\Lambda}_l(\mathbf{B}_k).
\end{aligned}
$$

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Summing these inequalities over k, we obtain

$$\mathrm{tr}\left(\boldsymbol{\Lambda}_l \mathbb{M}^{-1} \boldsymbol{\Lambda}_l\right) \geq \sum_{k=1}^{d_*} \int_\Omega \boldsymbol{\Lambda}_l(\mathbf{B}_k) : F_{\mathbb{C}^0}\left(1_\Omega \boldsymbol{\Lambda}_l(\mathbf{B}_k)\right) \tag{5.17}$$

$$+ \sum_{k=1}^{d_*} \int_\Omega \left(\mathbb{C}^0 - \mathbb{C}^1\right)^{-1} \boldsymbol{\Lambda}_l(\mathbf{B}_k) : \boldsymbol{\Lambda}_l(\mathbf{B}_k) + 2\mathrm{tr}\left(\boldsymbol{\Lambda}_l \mathbb{M}^{-1} \boldsymbol{\Lambda}_l\right).$$

It is proved in [9, (4.27) & (4.28)] that

$$\sum_{k=1}^{d_*} \int_\Omega \boldsymbol{\Lambda}_1(\mathbf{B}_k) : F_{\mathbb{C}^0}\left(1_\Omega \boldsymbol{\Lambda}_1(\mathbf{B}_k)\right) = -\frac{1}{d\left(\lambda + 2\mu\right)},$$

and

$$\sum_{k=1}^{d_*} \int_\Omega \boldsymbol{\Lambda}_2(\mathbf{B}_k) : F_{\mathbb{C}^0}\left(1_\Omega \boldsymbol{\Lambda}_2(\mathbf{B}_k)\right) = -\left(\frac{d-1}{d\left(\lambda + 2\mu\right)} + \frac{d-1}{2\mu}\right).$$

Since

$$\left(\mathbb{C}^0 - \mathbb{C}^1\right)^{-1} = \frac{1}{d(\kappa - \tilde{\kappa})}\boldsymbol{\Lambda}_1 + \frac{1}{2(\mu - \tilde{\mu})}\boldsymbol{\Lambda}_2,$$

one can immediately see that

$$\sum_{k=1}^{d_*} \int_\Omega \left(\mathbb{C}^0 - \mathbb{C}^1\right)^{-1} \boldsymbol{\Lambda}_1(\mathbf{B}_k) : \boldsymbol{\Lambda}_1(\mathbf{B}_k) = \frac{1}{d(\kappa - \tilde{\kappa})}$$

and

$$\sum_{k=1}^{d_*} \int_\Omega \left(\mathbb{C}^0 - \mathbb{C}^1\right)^{-1} \boldsymbol{\Lambda}_2(\mathbf{B}_k) : \boldsymbol{\Lambda}_2(\mathbf{B}_k) = \frac{d_* - 1}{2(\mu - \tilde{\mu})}.$$

Therefore, we get

$$\sum_{k=1}^{d_*} \left[\int_\Omega \boldsymbol{\Lambda}_l(\mathbf{B}_k) : F_{\mathbb{C}^0}\left(1_\Omega \boldsymbol{\Lambda}_l(\mathbf{B}_k)\right) + \int_\Omega \left(\mathbb{C}^0 - \mathbb{C}^1\right)^{-1} \boldsymbol{\Lambda}_l(\mathbf{B}_k) : \boldsymbol{\Lambda}_l(\mathbf{B}_k)\right] = -K_l$$

for $l = 1, 2$, where K_l is given in (1.8) and (1.9). It then follows from (5.17) that

$$\mathrm{tr}\left(\boldsymbol{\Lambda}_l \mathbb{M}^{-1} \boldsymbol{\Lambda}_l\right) \geq -K_l + 2\mathrm{tr}\left(\boldsymbol{\Lambda}_l \mathbb{M}^{-1} \boldsymbol{\Lambda}_l\right). \tag{5.18}$$

Suppose that equality in (1.10) holds. Then, in view of (5.18), the inequality in (5.17) becomes an equality, and so does the one in (5.16). Since $\boldsymbol{\Lambda}_1(\mathbf{B}_1) = \mathbf{B}_1$ and $\boldsymbol{\Lambda}_1(\mathbf{B}_k) = 0$ for $k = 2, \ldots, d_*$, we have

$$E_\mathbf{A}(\mathbb{C}^0, 1_\Omega \mathbf{B}_1) = \sup_{\mathbf{P} \in L^2(\mathbb{R}^d : M_d^s)} E_\mathbf{A}(\mathbb{C}^0, \mathbf{P}), \quad \mathbf{A} = \mathbb{M}^{-1}\mathbf{B}_1. \tag{5.19}$$

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Likewise, if equality in (1.11) holds, then

$$E_{\mathbf{A}}(\mathbb{C}^0, 1_\Omega \mathbf{B}_k) = \sup_{\mathbf{P} \in L^2(\mathbb{R}^d : M_d^s)} E_{\mathbf{A}}(\mathbb{C}^0, \mathbf{P}), \quad \mathbf{A} = \mathbb{M}^{-1} \mathbf{B}_k, \tag{5.20}$$

for $k = 2, \ldots, d_*$, and the proof is complete. $\qquad\square$

A   Proof of Lemma 4.1

Proof.   By considering $\frac{1}{t}\mathbf{B}_1 + \mathbf{B}_2$ and taking the limit $t \to \infty$, one can see that $\mathbf{B}_2$ also has a multiple eigenvalue. If $\mathbf{B}_2$ has an eigenvalue of multiplicity 3, then $\mathbf{B}_2$ is a constant multiple of the identity matrix and hence the conclusion of the lemma holds trivially.

Let us assume that $\mathbf{B}_2$ has an eigenvalue of multiplicity 2. Note that, for any real number s and orthogonal matrix $\mathbf{U}$,

$$\mathbf{U}\mathbf{B}_1 \mathbf{U}^{-1} + t\mathbf{U}\mathbf{B}_2 \mathbf{U}^{-1} - s\mathbf{I} = \mathbf{U}(\mathbf{B}_1 + t\mathbf{B}_2 - s\mathbf{I})\mathbf{U}^{-1}$$

has a multiple eigenvalue regardless of t. Therefore we may assume that $\mathbf{B}_2$ takes the form

$$\mathbf{B}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

while $\mathbf{B}_1$ is arbitrary, say

$$\mathbf{B}_1 = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}.$$

Let $\Gamma(t)$ be the discriminant of the characteristic polynomial of $\mathbf{B}_1 + t\mathbf{B}_2$. Since $\mathbf{B}_1 + t\mathbf{B}_2$ has a multiple eigenvalue for all t, $\Gamma(t) \equiv 0$. Then a straightforward calculation shows that the coefficient of t^4 term of $\Gamma(t)$ is given by $a^2 + b^2 - 2ab + 4d^2$, and hence we have

$$a - b = d = 0.$$

It then follows that the coefficient of t^2 term of $\Gamma(t)$ is given by $(e^2 + f^2)^2$, and hence

$$e = f = 0.$$

Thus $\mathbf{B}_1$ takes the form

$$\mathbf{B}_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}.$$

This completes the proof. $\qquad\square$

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